Problem 1.5

Prove the **BAC–CAB** rule by writing out both sides in component form.

Solution

The BAC-CAB rule is

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}).$$

Evaluate the left-hand side.

$$\begin{aligned}
\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \mathbf{A}\times\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix}\\
&= \mathbf{A}\times[(B_yC_z - B_zC_y)\hat{\mathbf{x}} - (B_xC_z - B_zC_x)\hat{\mathbf{y}} + (B_xC_y - B_yC_x)\hat{\mathbf{z}}]\\
&= \mathbf{A}\times[(B_yC_z - B_zC_y)\hat{\mathbf{x}} + (B_zC_x - B_xC_z)\hat{\mathbf{y}} + (B_xC_y - B_yC_x)\hat{\mathbf{z}}]\\
&= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ A_x & A_y & A_z\\ B_yC_z - B_zC_y & B_zC_x - B_xC_z & B_xC_y - B_yC_x\end{vmatrix}\\
&= [A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)]\hat{\mathbf{x}}\\
&\qquad - [A_x(B_xC_y - B_yC_x) - A_z(B_yC_z - B_zC_y)]\hat{\mathbf{y}}\\
&\qquad\qquad + [A_x(B_zC_x - B_xC_z) - A_y(B_yC_z - B_zC_y)]\hat{\mathbf{z}}\\
&= (A_yB_xC_y - A_yB_yC_x - A_zB_zC_x + A_zB_xC_z)\hat{\mathbf{x}}\\
&\qquad + (-A_xB_xC_y + A_xB_yC_x + A_zB_yC_z - A_zB_zC_y)\hat{\mathbf{y}}\\
&\qquad\qquad + (A_xB_zC_x - A_xB_xC_z - A_yB_yC_z + A_yB_zC_y)\hat{\mathbf{z}}
\end{aligned}$$

Evaluate the right-hand side.

$$\begin{aligned}
\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) &= (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})(A_xC_x + A_yC_y + A_zC_z)\\
&\qquad - (C_x\hat{\mathbf{x}} + C_y\hat{\mathbf{y}} + C_z\hat{\mathbf{z}})(A_xB_x + A_yB_y + A_zB_z)\\
&= [B_x(A_xC_x + A_yC_y + A_zC_z)\hat{\mathbf{x}} + B_y(A_xC_x + A_yC_y + A_zC_z)\hat{\mathbf{y}}\\
&\quad + B_z(A_xC_x + A_yC_y + A_zC_z)\hat{\mathbf{z}}] - [C_x(A_xB_x + A_yB_y + A_zB_z)\hat{\mathbf{x}}\\
&\qquad + C_y(A_xB_x + A_yB_y + A_zB_z)\hat{\mathbf{y}} + C_z(A_xB_x + A_yB_y + A_zB_z)\hat{\mathbf{z}}]\\
&= [B_x(A_xC_x + A_yC_y + A_zC_z) - C_x(A_xB_x + A_yB_y + A_zB_z)]\hat{\mathbf{x}}\\
&\quad + [B_y(A_xC_x + A_yC_y + A_zC_z) - C_y(A_xB_x + A_yB_y + A_zB_z)]\hat{\mathbf{y}}\\
&\qquad + [B_z(A_xC_x + A_yC_y + A_zC_z) - C_z(A_xB_x + A_yB_y + A_zB_z)]\hat{\mathbf{z}}\\
&= (\cancel{A_xB_xC_x} + A_yB_xC_y + A_zB_xC_z - \cancel{A_xB_xC_x} - A_yB_yC_x - A_zB_zC_x)\hat{\mathbf{x}}\\
&\quad + (A_xB_yC_x + \cancel{A_yB_yC_y} + A_zB_yC_z - A_xB_xC_y - \cancel{A_yB_yC_y} - A_zB_zC_y)\hat{\mathbf{y}}\\
&\qquad + (A_xB_zC_x + A_yB_zC_y + \cancel{A_zB_zC_z} - A_xB_xC_z - A_yB_yC_z - \cancel{A_zB_zC_z})\hat{\mathbf{z}}\\
&= (A_yB_xC_y - A_yB_yC_x - A_zB_zC_x + A_zB_xC_z)\hat{\mathbf{x}}\\
&\qquad + (-A_xB_xC_y + A_xB_yC_x + A_zB_yC_z - A_zB_zC_y)\hat{\mathbf{y}}\\
&\qquad\qquad + (A_xB_zC_x - A_xB_xC_z - A_yB_yC_z + A_yB_zC_y)\hat{\mathbf{z}}
\end{aligned}$$

Both sides have equal components, so the rule is proven.